with $f_e(\tau)$ denoting the exact temperature, ε denoting the magnitude of the entered relative error, and δ denoting a random number generated by the random numbers generator to simulate the fluctuation errors of measurements. An analysis of these results confirms the effectiveness of the regularization algorithm and the reliability of the solution. A solution with extrapolation can, however, be recommended only for slowly evolving thermal processes. Increasing the relative error of input temperatures does not give rise to any other singularities in the solution to a reverse heat-conduction problem. As the analysis of results (Figs. 1 and 4) indicates, however, an inaccurate stipulation of the boundary condition for the regularizing spline gives rise to appreciable errors in the retrieved boundary functions within a finite time interval. In our case the condition $S_\Delta'''(b) = 0$ was stipulated for the spline, this being correct for steady-state thermal conditions. At the end of the interval, therefore, this condition should correspond to steady-state thermal conditions.

NOTATION

T, temperature; t, time coordinate; x, space coordinate; and a, thermal diffusivity.

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ESTIMATES OF THE VALIDITY RANGE FOR THE HYPERBOLIC EQUATION OF HEAT-CONDUCTION IN HOMOGENEOUS SYMMETRIC CONTINUOUS BODIES

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Estimates are made of the geometrical dimensions of symmetric continuous bodies, the temperature fields within which can be described by, respectively, the hyperbolic or the parabolic heat-conduction equation.

The phenomenological heat-conduction theory has been developed in a formulation uniform with respect to geometrical variables [1]. An analysis of heat and mass transfer, especially when the process is nonsteady and very intensive, leads to a hyperbolic heat-conduction equation and has served as a basis for a dynamic heat-conduction theory [2].

The problem of determining the structure of the temperature field in homogeneous symmetric continuous bodies reduces, within the dynamic theory of heat conduction, to that of finding within the region $D = [0, t_1] \times \Omega = \{(t, r), 0 \leqslant t \leqslant t_1, 0 \leqslant r \leqslant R\}$ a bounded and sufficiently smooth solution to the equation [2]

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$$\frac{1}{c_q^2}\frac{\partial^2 T}{\partial t^2}+\frac{1}{a}\frac{\partial T}{\partial t}-\left(\frac{\partial^2}{\partial r^2}+\frac{2\nu+1}{r}\frac{\partial}{\partial r}\right)T=f(t,r) \tag{1}$$

for the initial conditions

$$T|_{t=0}=g_1(r),\quad \left.\frac{\partial T}{\partial t}\right|_{t=0}=g_2(r) \tag{2}$$

and the boundary conditions

$$\lim_{r\to 0}\frac{\partial T}{\partial r}=0,\quad \left.\left(\frac{\partial T}{\partial r}+\beta T\right)\right|_{r=R}=\beta T_c(t)-\frac{1}{\lambda}\left(1+\tau_r\frac{\partial}{\partial t}\right)q_r(t)\equiv g_3(t). \tag{3}$$

We will introduce into the analysis the Hankel finite integral transformation $H_{\nu\nu}$ and the inverse finite integral transformation $(H_{\nu\nu})^{-1}$, both of the first kind, according to the rules

$$H_{\nu\nu}[f(r)]=\int_0^R f(r)J_{\nu,\nu}(\lambda_n r)r^{2\nu+1}dr\equiv f_n, \tag{4}$$

$$H_{\nu\nu}^{-1}[f_n]=\sum_{n=0}^{\infty}f_n\frac{J_{\nu\nu}(\lambda_n r)}{\|J_{\nu,\nu}(\lambda_n r)\|^2}\equiv f(r), \tag{5}$$

respectively, where λ_n are roots of the transcendental Bessel equation of the first kind

$$\lambda J_{\nu,\nu}^1(\lambda R)+\beta J_{\nu,\nu}(\lambda R)=0;\quad (\beta J_{\nu,\nu}(\lambda R)-R\lambda^2 J_{\nu+1,\,\nu+1}(\lambda R)=0), \tag{6}$$

and where

$$\|J_{\nu,\nu}(\lambda_n R)\|^2=\frac{1}{2}R^{2(\nu+1)}[J_{\nu,\nu}^2(\lambda_n R)+\lambda_n^2R^2J_{\nu+1,\,\nu+1}^2(\lambda_n R)] \tag{7}$$

is the norm squared, with $J_{\nu,\nu}(x) = x^{-\nu}J_\nu(x)$; and $J_\nu(x)$ is the Bessel function of the first order of argument ν [3].

Equation (6) indicates that, with a boundary condition of the second kind ($\beta = 0$) stipulated at the surface $r = R$, it has the root $\lambda = 0$ (eigenvalue). Then series (5) becomes

$$\frac{2}{R^{2(\nu+1)}}\left[f_0+\sum_{n=1}^{\infty}f_n\frac{J_{\nu,\nu}(\lambda_n r)}{J_{\nu,\nu}^2(\lambda_n R)}\right],\quad f_0=\int_0^R f(r)\,r^{2\nu+1}dr.$$

The operator $H_{\nu\nu}$ establishes a correspondence between problem (1)-(3) and the problem of integrating the ordinary differential equation

$$\frac{1}{c_q^2}\frac{d^2T_n}{dt^2}+\frac{1}{a}\frac{dT_n}{dt}+\lambda_n^2T_n=f_n(t)+R^{2\nu+1}J_{\nu,\nu}(\lambda_n R)\,g_3(t)$$

for the initial conditions

$$T_n|_{t=0}=g_{1n},\quad \left.\frac{dT_n}{dt}\right|_{t=0}=g_{2n}.$$

Letting

$$T_n(t)=A_n(t)+g_{1n}+tg_{2n}, \tag{8}$$

we obtain for the new sought function $A_n(t)$ The problem

$$\frac{d^2A_n}{dt^2}+\frac{c_q^2}{a}\frac{dA_n}{dt}+c_q^2\lambda_n^2A_n(t)=g_n(t)\,c_q^2, \tag{9}$$

$$A_n(0)=0,\quad \frac{dA_n(0)}{dt}=0, \tag{10}$$

with the functions

$$g_n(t)=f_n(t)+R^{2\nu+1}J_{\nu,\nu}(\lambda_\nu R)\,g_3(t)-\frac{1}{a}g_{2n}-\lambda_n^2(g_{1n}+tg_{2n}).$$

Since the Cauchy function in problem (9)-(10) is [4]

$$K_n(t-\tau)=\begin{cases}(t-\tau)\exp\left(-\frac{t-\tau}{2\tau_r}\right), & \text{when } \delta_n=0,\\ \frac{2\tau_r}{\sqrt{\delta_n}}\exp\left(-\frac{t-\tau}{2\tau_r}\right)\operatorname{sh}\frac{\sqrt{\delta_n}}{2\tau_r}(t-\tau), & \text{when } \delta_n\neq 0,\end{cases} \quad (11)$$

$$\delta_n=1-4\lambda_n^2 a\tau_r=1-4\lambda_n^2 a^2 c_q^{-2},$$

hence the solution to problem (9)-(10) will be the function

$$A_n(t)=\begin{cases}a\int_0^t\frac{t-\tau}{2\tau_r}\exp\left(-\frac{t-\tau}{2\tau_r}\right)g_n(\tau)\,d\tau \text{ when } \delta_n=0,\\ \frac{2a}{\sqrt{\delta_n}}\int_0^t\exp\left(-\frac{t-\tau}{2\tau_r}\right)\operatorname{sh}\frac{\sqrt{\delta_n}}{2\tau_r}(t-\tau)\,g_n(\tau)\,d\tau \text{ when } \delta_n\neq 0.\end{cases} \quad (12)$$

By virtue of Eq. (9), moreover, we have the identity

$$\left(\frac{d^2}{dt^2}+\frac{c_q^2}{a}\frac{d}{dt}\right)K_n(t)\equiv -c_q^2\lambda_n^2 K_n(t), \quad (13)$$

and by virtue of the properties of function $J_{\nu,\nu}$ the identities

$$B_\nu[J_{\nu,\nu}(\lambda r)]\equiv\left(\frac{\partial^2}{\partial r^2}+\frac{2\nu+1}{r}\frac{\partial}{\partial r}\right)J_{\nu,\nu}(r\lambda)\equiv -\lambda^2 J_{\nu,\nu}(\lambda r),$$

$$H_{\nu,\nu}[B_\nu(f(r))]\equiv -\lambda_n^2 f_n+R^{2\nu+1}\left(\frac{\partial f}{\partial r}J_{\nu,\nu}(\lambda_n r)-f(r)\frac{\partial J_{\nu,\nu}(\lambda_n r)}{\partial r}\right)\Bigg|_{r=R}.$$

Application of operator $H_{\nu,\nu}^{-1}$ to expression (8) according to rule (5) yields the solution to the original problem, which can be put in the form

$$T(r,t)=R_{ph}(r,t)+R_0(r,t)+R_h(r,t)+R_{m_0}(r,t),$$

where

$$R_{ph}(r,t)=\sum_{n=0}^{n_0-1}A_n(t)X_n(r)+\sum_{n=0}^{\infty}g_{1n}(r)X_n(r)$$

for all n for which $\delta_n > 0$, and $R_0(r, t) = A_{n_0}(t)X_{n_0}(r)$ when there exists an n_0 such that $\delta_{n_0} = 0$, and

$$R_h(r,t)=\sum_{n=n_0+1}^{m_0}A_n(t)X_n(r)+\sum_{n=0}^{\infty}tg_{2n}(r)X_n(r)$$

for all $n\leqslant m_0$ for which $\delta_n < 0$ (R_{ph}, R_0, and R_h, denoting, respectively, the parabolic—hyperbolic part, the transitional part, and the hyperbolic part of the solution to problem (1)-(3)), and

$$R_{m_0}(r,t)=\sum_{n=m_0+1}^{\infty}A_n(t)X_n(r)$$

is the remainder of the series with number m_0 defined by the condition

$$\sum_{n=m_0+1}^{\infty}|A_n(t)X_n(r)|<\varepsilon_0$$

for all $t \in [0, \infty)$, and with ε_0 denoting an arbitrarily small positive number so that

$$X_n(r)=\frac{J_{\nu,\nu}(\lambda_n r)}{\|J_{\nu,\nu}(\lambda_n r)\|^2}.$$

We will use function (11) as the basis of the validity criterion for describing the thermal state of symmetric continuous bodies with a radius R by the hyperbolic heat-conduction equation.

For boundary-value problems I and II, the transcendental Bessel equation (6) becomes, respectively,

$$J_{\nu,\nu}(\lambda R)=0, \quad \lambda J'_{\nu,\nu}(\lambda R)=0.$$

The roots of these equations are [5]

$$\lambda_n^{\mathrm{I}}=\frac{b_1(n)}{R}; \quad \lambda_n^{\mathrm{II}}=\frac{b_2(n)}{R}, \tag{14}$$

where

$$b_1(n)=\left[\beta^1-\frac{m-1}{8\beta^1}-\frac{4(m-3)(7m-3)}{3(8\beta^1)^3}-\frac{32(m-1)(83m^2-982m-3779)}{15(8\beta^1)^5}-\right.$$
$$\left.-\frac{64(m-1)(6949m^3-153855m^2-1585743m-6277237)}{105(8\beta^1)^7}-\ldots\right];$$

$$b_2(n)=\left[\gamma-\frac{m+3}{8\gamma}-\frac{4(7m^2-82m-9)}{3(8\gamma)^2}-\frac{32(83m^2-2075m^2-3039m+3537)}{15(8\gamma)^3}-\ldots\right];$$

$$\beta^1=\frac{1}{4}\pi(2\nu+4n-1); \quad \gamma=\frac{1}{4}\pi(2\nu+4n+1);$$

$$m=4\nu^2 \quad \text{when} \quad -\frac{1}{2}<\nu<\frac{1}{2};$$

$$b_1(n)=l_1\pi-\frac{4\nu^2-1}{8\pi l_1}-\frac{(4\nu^2-1)(28\nu^2-31)}{384(\pi l_1)^3}-\ldots;$$

$$b_2(n)=\frac{1}{2}\left[l_2\pi-\frac{\nu^2-2\nu}{2l_2}-\frac{(4\nu^2-8\nu)(28\nu^2-56\nu-30)}{384(\pi l_2)^3}-\ldots\right];$$

$$l_1=n+\frac{1}{2}\nu-\frac{1}{4}; \quad l_2=n+\frac{1}{2}\nu-\frac{3}{4} \quad \text{when } \nu>\frac{1}{2}.$$

Let $\delta_n > 0$ for all n. Then

$$c_q^2>4\lambda_n^2a^2 \tag{15}$$

for all n = 1, 2, 3, ... and, inasmuch as $\lambda_n=(\lambda_n^{\mathrm{I}}, \lambda_n^{\mathrm{II}}) \to \infty$ when $n \to \infty$, obviously inequality (15) holds true when $c_q = \infty$. Accordingly, condition (15) is satisfied within the scope of the phenomenological heat-conduction theory. Here $R_h = R_{m_0} = 0$.

Let $\delta_n > 0$ for all $n \leqslant m_0$. Then

$$c_q^2>4\lambda_{m_0}^2a^2. \tag{16}$$

When heat propagates through a body at a velocity which satisfies inequality (16), then, allowing an error ε_0 for all $t \in [0, \infty)$, one can describe the process by the classical heat-conduction equation.

Estimate (16) indicates also that, at any arbitrarily high but finite velocity of heat propagation through a body, the solution to problem (1)-(3) will always contain a hyperbolic part.

Let $\delta_n < 0$ for all n. Then, taking into account the properties of the roots of Bessel functions $J_\nu(x)$ and $J'_\nu(x)$ (with $\nu > -1$), we can write

$$c_q^2<4\lambda_1^2a^2. \tag{17}$$

When heat propagates through a body at a velocity which satisfies inequality (17), then the hyperbolic equation must be used for calculating the temperature field in the body and, moreover, $R_0 \equiv 0$. Estimate (17), with roots (14) for all n, yields the estimate for the geometrical dimension of the region

$$0<\max R_\nu^{\mathrm{I}}<\frac{2b_1(n)}{c_q}a \quad \text{and} \quad 0<\max R_\nu^{\mathrm{II}}<\frac{2b_2(n)}{c_q}a$$

where the thermal process corresponding to, respectively, boundary-value problems I and II is describable by the hyperbolic equation. For bodies with axial symmetry, where $b_1(1) = 2.4048856$ and $b_2(1) = 3.8317060$ [6], the size of this dimension is, respectively,

$$0 < \min R_0^{\mathrm{I}} < c \cdot 10^{-6}\ \mathrm{m}; \quad 0 < \min R_0^{\mathrm{II}} < c_1 \cdot 10^{-6}\ \mathrm{m}.$$

The values of constants c and c_1 depend on the material. For aluminum, copper, steel, iron, gold, and cork they are 0.14576 and 0.23225, 0.1622 and 0.2584, 0.05185 and 0.08253, 0.06713 and 0.10695, 0.1746 and 0.2782, 0.43026 and 0.68553, respectively. Quite evidently, the region within which the thermodynamic equilibrium stabilizes is wider under the conditions of boundary-value problem II than under the conditions of boundary-value problem I.

For bodies with central symmetry, where $\beta \to \infty$, Eq. (6) with the Bessel function $J_\nu(x)$ expressed through algebraic and trigonometric functions of x [3] can be written as

$$\sqrt{\frac{2}{\pi}}\,\frac{\sin\lambda R_{1/2}^{\mathrm{I}}}{\lambda R_{1/2}^{\mathrm{I}}} = 0.$$

Then $\lambda_n^{\mathrm{I}} = n\pi/R_{1/2}^{\mathrm{I}}$ and condition (17) yields the estimate

$$\min R_{1/2}^{\mathrm{I}} < \frac{a}{c_q}\,2\pi,$$

indicating that in this case the width of the region where the hyperbolic equation (1) for $\nu = 1/2$ describes the thermal process is equal to the width of the region [7] where the equation

$$\frac{1}{c_q^2}\frac{\partial^2 T}{\partial t^2} + \frac{1}{a}\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial r^2} = f(t,\ r)$$

with initial conditions (2) and boundary conditions of the first or the second kind describes the temperature distribution.

When $\beta = 0$, then the eigenvalues λ_n^{II} are determined from the transcendental equation

$$\sqrt{\frac{2}{\pi}}\left(\frac{1}{\lambda_n^{\mathrm{II}}R_{1/2}^{\mathrm{II}}}\right)^2\left(\frac{\sin\lambda_n^{\mathrm{II}}R_{1/2}^{\mathrm{II}}}{\lambda_n^{\mathrm{II}}R_{1/2}^{\mathrm{II}}} - \cos\lambda_n^{\mathrm{II}}R_{1/2}^{\mathrm{II}}\right) = 0,$$

and $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$. The estimate for the geometrical dimension is then

$$\max R_{1/2}^{\mathrm{II}} < \frac{2\lambda_n^{\mathrm{II}}a}{c_q}.$$

Using $\lambda_1^{\mathrm{II}} = 1.16238$ [1], we have calculated min $R_{1/2}^{\mathrm{II}}$ for aluminum, copper, steel, and iron, for which it does not exceed $0.07045 \cdot 10^{-6}$, $0.0783 \cdot 10^{-6}$, $0.02503 \cdot 10^{-6}$, and $0.03244 \cdot 10^{-6}$ m, respectively. An analysis of the values obtained for $R_{1/2}^{\mathrm{II}}$ and $R_{1/2}^{\mathrm{I}}$ [7] reveals that the region within which the thermodynamic equilibrium stabilizes is now narrower under the conditions of boundary-value problem II than under the condition of boundary-value problem I.

Relation (15) indicates that in the case of a thermal process describable by the parabolic heat-conduction equation ($c_q = \infty$) the region is as wide as the entire body. From condition (16) follows

$$R_\nu^{\mathrm{I}} > \frac{2b_1(m_0)a}{c_q} \equiv R_1; \quad R_\nu^{\mathrm{II}} > \frac{2b_2(m_0)a}{c_q} \equiv R_2,$$

i.e., the classical equation of heat conduction applies, within an accuracy down to ε_0, to bodies with geometrical dimensions no smaller than R_1 and R_2. When $R_\nu^{\mathrm{I}} < R_1$ or $R_\nu^{\mathrm{II}} < R_2$ in boundary-value problems I and II, respectively, then thermal processes in symmetric continuous bodies must be described by the hyperbolic equation.

The transitional part of problem (1)-(3) appears when number n_0 is such that $c_q = 2\lambda_{n_0}a$, i.e., c_q satisfies the equation

$$\frac{c_q}{2a}J'_{\nu,\nu}\left(\frac{c_qR}{2a}\right) + \beta J_{\nu,\nu}\left(\frac{c_qR}{2a}\right) = 0.$$

According to estimates (15)-(17), in establishing the validity of either the classical or the hyperbolic heat-conduction equation for symmetric continuous bodies, one must take into account not only the velocity of heat propagation but also the mode of interaction between the body and the ambient medium.

NOTATION

$c_q = \sqrt{a/\tau_r}$, velocity of heat propagation; a, thermal diffusivity; λ, thermal conductivity; τ_r, thermal flux relaxation time; $T_c(t)$, ambient temperature at the surface $r = R$; $q_r(t)$, thermal flux in the radial direction at the surface; β, coupling coefficient characterizing the boundary conditions of the first or the second kind ($\beta = 0$ in the second kind, $\beta \to \infty$ in the first kind); and $2\nu + 1 \geqslant 0$: with $\nu = 0$ we have axial (cylindrical) symmetry and with $\nu = {}^1/_2$ we have central (spherical) symmetry.

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